Eur. Phys. J. C 1, 721–728 (1998)

THE EUROPEAN
PHYSICAL JOURNAL C

Reconstruction of SU(5) grand unified model in noncommutative geometry approach

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Received: 12 January 1997 / Revised version: 9 May 1997

Abstract. Based on a generalized gauge theory on $M^4 \times Z_2 \times Z_3$, we reconstruct the realistic SU(5) grand unified model by a suitable assignment of fermion fields. The action of group elements Z_2 on fermion fields is the charge conjugation, and the action of Z_3 elements represent generation translation. We find that a linear term of curvature has to be introduced to accommodate the spontaneous symmetry breaking and gauge hierarchy of the SU(5) model. A new mass relation is obtained in our reconstructed model.

1 Introduction

In recent years, it is believed that non-commutative geometry extends the basic geometry framework of physics [1,2]. The most remarkable results are that in the standard model the Higgs fields may be considered as a kind of gauge field on the same footing as Yang-Mills fields, and the Yukawa couplings can be introduced as a kind of gauge coupling. These topics have been studied by many authors [3]–[10]. It is also interesting to ask whether the same description stands when we go from the standard model to the grand unification theories (e.g. SU(5) GUT [12]), in which Higgs fields are introduced as input data in the model building. By enlarging the discrete points model first proposed by A. Connes [1,3], A. Chamsedine et al. [5] provided a generalized formula, which gave a clue for how to study more extensive models beyond the standard model, such as the SU(5) and SO(10) grand unified models. However, lots of details need to be further studied.

In our previous work [7,8], we constructed a generalized gauge theory of the discrete group Z_2. In this approach, we enlarged space–time to five dimensions with the 5th "coordinate" consisting of only two points of Z_2, assigned left- and right-handed Fermion fields according to the discrete group "coordinate", and wrote down a Lagrangian for the fermion fields, which is not only the function of the space–time coordinates but also of the discrete group "coordinate". The most important point of this approach was that the derivatives on discrete group were included in the Lagrangian. As for the case of the ordinary Yang–Mills gauge theory, when we require the Lagrangian be invariant under the action of the gauge group that is a function of space–time and of the discrete group, the Higgs fields appear in the covariant derivative and Yukawa coupling is naturally introduced by the gauge coupling. Furthermore, we constructed the Weinberg–Salam model and the electroweak–strong interaction model and tried to endow the discrete group with some physical meaning.

In this paper, we first develop our previous approach to the case of $M^4 \times Z_2 \times Z_3$ and reconstruct the realistic SU(5) grand unified model of three-generation fermions with the generalized gauge theory on $M^4 \times Z_2 \times Z_3$. A similar generalized gauge theory on $M^4 \times Z_2 \times Z_3$ has also been discussed in a CP-violation toy model [11]. We distinguish the left- and right-hand parts of fermions by two elements of the discrete group Z_2, differentiate three families by three elements of the discrete group Z_3, and connect fermions by charge conjugation transformation on discrete points of Z_2 and by generation translation on discrete points of Z_3. Since there are two mass scales in the SU(5) model characterizing the spontaneous symmetry breaking of SU(5) to SU(3) × SU(2) × U(1) and then to SU(3) × U(1), if we want to get this gauge hierarchy, we need to add the linear term of curvature F, first proposed by Sitarz [6].

The plan of this paper is as follows. In Sect. 2, we review gauge theory on $M^4 \times Z_2 \times Z_3$. In Sect. 3, we build the SU(5) model using a generalized gauge theory on $M^4 \times Z_2 \times Z_3$. In Sect. 4, we discuss the symmetry breaking phenomenon.

2 Notation of gauge theory on $M^4 \times Z_2 \times Z_3$

In this section we give a basic review of gauge theory on $M^4 \times Z_2 \times Z_3$. A more detailed account of this construction may be found in [6,7].

Let x^μ denote the coordinate on M^4 and g label the points of the discrete group $Z_2 \times Z_3$. The differentiation of an arbitrary function on product space $M^4 \times Z_2 \times Z_3$

has the following form

$$\mathrm{d}f = \partial_\mu f \mathrm{d}x^\mu + \partial_g f \chi^g, g \in Z_2 \times Z_3, \tag{2.1}$$

where $\mathrm{d}x^\mu$ and χ^g are basis of one forms on M^4 and $Z_2 \times Z_3$, respectively. The partial derivative ∂_g is defined as

$$\begin{aligned}\partial_g f(x,h) &= (f(x,h) - R_g f(x,h)) \\ &= (f(x,h) - f(x, h\cdot g)).\end{aligned} \tag{2.2}$$

From this definition we may obtain a lot of relations: some for the product of one-forms,

$$\begin{aligned}\mathrm{d}x^\mu \hat{\otimes} \mathrm{d}x^\nu &= -\mathrm{d}x^\nu \hat{\otimes} \mathrm{d}x^\mu \\ \mathrm{d}x^\mu \hat{\otimes} \chi^g &= -\chi^g \hat{\otimes} \mathrm{d}x^\mu,\end{aligned} \tag{2.3}$$

some for the multiplication of one-forms by functions,

$$\begin{aligned}f(x,h)\mathrm{d}x^\mu &= \mathrm{d}x^\mu f(x,h), \\ \chi^g f(x,h) &= R_g f(x,h)\chi^g,\end{aligned} \tag{2.4}$$

and some for the action of derivative operators on one-forms

$$\mathrm{d}\mathrm{d}x^\mu = 0,\ \mathrm{d}\chi^g = -C^g_{p,h}\chi^p \hat{\otimes} \chi^h,$$

where the structure constants are given by $C^g_{p,h} = \delta^g_p + \delta^g_h - \delta^g_{ph}(\delta^e_{ph} - 1)$. The general gauge potential A on $M^4 \times Z_2 \times Z_3$ may be written as

$$A = A_\mu \mathrm{d}x^\mu + \sum_{\substack{g \in Z_2 \times Z_3 \\ g \neq e}} \phi_g \chi^g \ . \tag{2.5}$$

The unitarity of the gauge group enforces that $A^* = -A$. Thus, since $(\mathrm{d}x^\mu)^* = \mathrm{d}x^\mu$ and $(\chi^g)^* = -\chi^{g^{-1}}$, we obtain

$$(A_\mu)^\dagger = -A_\mu,\ \phi^\dagger_g = R_g \phi_{g^{-1}} \ .$$

The curvature two-form $F = \mathrm{d}A + A \hat{\otimes} A$ splits into

$$F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \hat{\otimes} \mathrm{d}x^\nu + F_{\mu g}\mathrm{d}x^\mu \hat{\otimes} \chi^g + F_{gh}\chi^g \hat{\otimes} \chi^h, \tag{2.6}$$

where

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\ , \\ F_{\mu g} &= \partial_\mu \Phi_g + A_\mu \Phi_g - \Phi_g R_g(A_\mu)\ , \\ F_{gh} &= \partial_g \phi_h + \phi_g R_g \phi_h - C^k_{gh}\phi_k\end{aligned} \tag{2.7}$$

with $\Phi_g = 1 - \phi_g$.

To construct the Yang–Mills action, we need to define the metric[1]

$$\begin{aligned}&\langle \mathrm{d}x^\mu, \mathrm{d}x^\nu \rangle = g^{\mu\nu},\quad \langle \chi^g, \chi^h \rangle = \eta^{gh}, \\ &\langle \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\rho \rangle = \tfrac{1}{2}(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}), \\ &\langle \mathrm{d}x^\mu \otimes \chi^g, \mathrm{d}x^\nu \otimes \chi^h \rangle = g^{\mu\nu}\eta^{gh}, \\ &\langle \chi^g \otimes \chi^h, \chi^{g'} \otimes \chi^{h'} \rangle = \eta^{gg'}\eta^{hh'}.\end{aligned} \tag{2.8}$$

[1] For the sake of simplicity, we have set the mass dimension equal to 1 in the mathematical formulas

where $\eta^{gh} = \eta_g \delta^{gh^{-1}}$. After such a form of the metric is taken, the Yang–Mills Lagrangian becomes

$$\begin{aligned}\mathcal{L}_\mathrm{N} &= -\frac{1}{N}\int_G \langle F, \overline{F} \rangle \\ &= \frac{1}{N}\int_G \left(-\frac{1}{4}F_{\mu\nu}F^{\dagger\mu\nu} + \eta_g F_{\mu g}F^{\dagger\mu}_g - \eta_g\eta_h F_{gh}F^\dagger_{gh} \right)\end{aligned} \tag{2.9}$$

It is possible to add an extra gauge-invariant term to the Yang–Mills action [6], which is linear in the curvature $\langle F \rangle$,

$$\begin{aligned}\mathcal{L}_\mathrm{L} &= -\frac{1}{N}\int_G \langle F \rangle \\ &= -\frac{1}{N}\int_G F_{gh}\eta^{gh} = -\frac{1}{N}\int_G \eta_g F_{gg^{-1}}.\end{aligned} \tag{2.10}$$

Let us add this term to the Yang–Mills action with an arbitrary scaling parameter α. We obtain the bosonic sector Lagrangian

$$\mathcal{L} = \mathcal{L}_\mathrm{N} + \alpha \mathcal{L}_\mathrm{L}. \tag{2.11}$$

In next section, we find that this Lagrangian is needed in the construction of the SU(5) model.

3 Generalized SU(5) gauge theory on $M^4 \times Z_2 \times Z_3$

In this section, we construct the SU(5) gauge theory on $M^4 \times Z_2 \times Z_3$ by using generalized gauge theory on the discrete group [7,8]. To this end, we first set fermion fields on the discrete group, then write down gauge fields in terms of the gauge potential. Finally, the Lagrangian of the gauge fields is written down via the noncommutative differential geometry approach.

3.1 Fields on $M^4 \times Z_2 \times Z_3$

From the basic knowledge of SU(5) model [14], we know that one family of left-handed (or right-handed) fermions can be accommodated in an SU(5) reducible representation of $5^* + 10$ (or $5 + 10^*$). According to the representation of SU(5), we write down the first family fermions as

$$5^*: \quad \psi_{\rm L} = \begin{bmatrix} d_1^C \\ d_2^C \\ d_3^C \\ e^- \\ -\nu_e \end{bmatrix}_{\rm L}, \quad 5: \psi_{\rm R}^C = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ e^+ \\ -\nu_e^C \end{bmatrix}_{\rm R}$$

$$10: \quad \chi_{\rm L} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u_3^C & -u_2^C & u_1 & d_1 \\ -u_3^C & 0 & u_1^C & u_2 & d_2 \\ u_2^C & -u_1^C & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^+ \\ -d_1 & -d_2 & -d_3 & -e^+ & 0 \end{bmatrix}_{\rm L}, \quad (3.1)$$

$$10^*: \chi_{\rm R}^C = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & u_3 & -u_2 & u_1^C & d_1^C \\ -u_3 & 0 & u_1 & u_2^C & d_2^C \\ u_2 & -u_1 & 0 & u_3^C & d_3^C \\ -u_1^C & -u_2^C & -u_3^C & 0 & e^- \\ -d_1^C & -d_2^C & -d_3^C & -e^- & 0 \end{bmatrix}_{\rm R}.$$

The other two families can be written similarly by replacing u, d, e, ν_e by c, s, μ, ν_μ and t, b, τ, ν_τ. From observation of three-family fermions and their left- and right-hand parts, $5^* + 10$ and $5 + 10^*$, we find it is possible to assign them with respect to elements of discrete group $Z_2 \times Z_3$, which is the product of discrete groups Z_2 and Z_3. The discrete group Z_2 has two elements $Z_2 = \{e, Z | Z^2 = e\}$; discrete group Z_3 has three elements $Z_3 = \{e, r, r^2 | r^3 = e\}$. So the direct product group $Z_2 \times Z_3$ has six elements:

$$Z_2 \times Z_3 = \{e, r, r^2 Z, Zr, Zr^2 | Z^2 = e, r^3 = e, Zr = rZ\}.$$

In this paper, we use two elements of Z_2 to distinguish left-right hand fermions and use three elements of Z_3 to distinguish three families. So the manifold considered should be $M^4 \times Z_2 \times Z_3$.

According to discrete group $Z_2 \times Z_3$, we arrange fermions as follows:

$$\begin{aligned} &\psi(x,e) = \begin{bmatrix} \psi^C \\ \chi^C \end{bmatrix}_{\rm R}^1, \quad \psi(x,r) = \begin{bmatrix} \psi^C \\ \chi^C \end{bmatrix}_{\rm R}^2, \\ &\psi(x,r^2) = \begin{bmatrix} \psi^C \\ \chi^C \end{bmatrix}_{\rm R}^3, \\ &\psi(x,Z) = \begin{bmatrix} \psi \\ \chi \end{bmatrix}_{\rm L}^1, \quad \psi(x,rZ) = \begin{bmatrix} \psi \\ \chi \end{bmatrix}_{\rm L}^2, \\ &\psi(x,r^2 Z) = \begin{bmatrix} \psi \\ \chi \end{bmatrix}_{\rm L,}^3 \end{aligned} \quad (3.2)$$

where $[]^i$ represents the ith generation of fermions. It is important to note that the actions $R_g, g \in Z_2 \times Z_3$ on fermions have definite physical meanings. We find that the action R_Z is nothing but the charge conjugation transformation, which changes left- and right-handed fermions between $5^* + 10$ and $5 + 10^*$ and the action $R_{r^i}, i = 1, 2, 3$ is the translation among different generations,

$$R_{r^i} \psi^j = \psi^{[(i+j)|Mod\ 3]}, \quad i = 1, 2; \ j = 1, 2, 3$$

As we did in [7,8], we build gauge theory on space $M^4 \times Z_2 \times Z_3$ by introducing the free fermion Lagrangian first,

$$\begin{aligned} \mathcal{L}(x,g) &= \overline{\psi}(g) \Big[i\gamma^\mu (\overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu) \\ &\quad -U(\partial_Z + \partial_{Zr} + \partial_{Zr^2}) - U_1(\partial_r + \partial_{r^2}) \Big] \psi(g), \\ &\quad g \in Z_2 \times Z_3 \end{aligned} \quad (3.3)$$

where U, U_1 are parameters with mass dimension. Because the parameters in front of the partial derivatives of the discrete group are directly related to the mass of Higgs particles in the reconstructed model and there are only two mass scales of Higgs fields in the minimum SU(5) model, we just choose two free parameters in front of the partial derivatives of the discrete group here. In fact, U and U_1 are parameters related to the distance among discrete points in the non-commutative geometry approach.

As with the introduction of Yang–Mills fields, it is reasonable to require that the Lagrangian (3.3) be invariant under gauge transformations $H(x,g), g \in Z_2 \times Z_3$, where H are functions depending not only on M^4 but also on the discrete group. So one should introduce a covariant derivative in the Lagrangian given in (3.3). The gauge-invariant Lagrangian under SU(5) group should be written as

$$\begin{aligned} &\mathcal{L}_{\rm F}(x,g) = \overline{\psi}(g) \Big[i\gamma^\mu (\overrightarrow{D}_\mu - \overleftarrow{D}_\mu) \\ &-U(D_Z + D_{Zr} + D_{Zr^2}) - U_1(D_r + D_{r^2}) \Big] \psi(g), \\ &g \in Z_2 \times Z_3, \end{aligned} \quad (3.4)$$

where $D_\mu = \partial_\mu + igA_\mu, \quad D_g = \partial_g + \phi_g R_g$ and

$$\begin{aligned} A(e) &= \begin{bmatrix} (A_{k,l}) & \\ & (A^*_{mn,pq}) \end{bmatrix}, \\ A(Z) &= \begin{bmatrix} \left(A^*_{k,l}\right) & \\ & (A_{mn,pq}) \end{bmatrix}; \end{aligned} \quad (3.5)$$

$(A_{k,l})$ is a 5×5 matrix valued on 24 generators of the SU(5) group and the corresponding matrix elements are $A_{k,l}$; $(A_{mn,pq})$ is a 25×25 matrix with mn, pq denoting the row and column indices of the matrix, and the matrix elements are

$$A_{mn,pq} = A_{m,p}\delta_{n,q} + A_{n,q}\delta_{m,p}.$$

Because the gauge transformations are independent of generations, we should set the Yang–Mills potentials to be the same in different generations. This means $A(e) = A(r) = A(r^2)$ and $A(Z) = A(rZ) = A(r^2 Z)$.

In the minimal SU(5) model, there are two Higgs multiplets, which belong to the adjoint and the vector representations, respectively. Only the vector Higgs field appears in Yukawa coupling. In Yukawa terms of the Lagrangian (3.4), it is easy to find that ϕ_Z, ϕ_{rZ}, ϕ_{r^2Z} connect left- and right-handed fermions and ϕ_r,ϕ_{r^2} connect fermions with the same chirality. So only ϕ_Z, ϕ_{rZ}, ϕ_{r^2Z} fields appear in Yukawa terms; ϕ_r,ϕ_{r^2} fields do not. To get the minimal SU(5) model, we arrange vector representation in ϕ_Z, ϕ_{rZ}, ϕ_{r^2Z} and adjoint representation in ϕ_r, ϕ_{r^2}. Thus we write down the fields $\phi_Z, \phi_{rZ}, \phi_{r^2Z}$ as

$$
\begin{aligned}
& \qquad\qquad g=e \\
\phi_Z(g) \;=\; & \begin{bmatrix} 0 & f_{11}\left(H^*_{i,mn}\right) \\ f_{11}\left(H^*_{pq,j}\right) & e_{11}\left(H_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r \\
& \begin{bmatrix} 0 & f_{22}\left(H^*_{i,mn}\right) \\ f_{22}\left(H^*_{pq,j}\right) & e_{22}\left(H_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r^2 \\
& \begin{bmatrix} 0 & f_{33}\left(H^*_{i,mn}\right) \\ f_{33}\left(H^*_{pq,j}\right) & e_{33}\left(H_{pq,mn}\right) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& \qquad\qquad g=e \\
\phi_{rZ}(g) \;=\; & \begin{bmatrix} 0 & f_{21}\left(H^*_{i,mn}\right) \\ f_{12}\left(H^*_{pq,j}\right) & e_{12}\left(H_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r \\
& \begin{bmatrix} 0 & f_{32}\left(H^*_{i,mn}\right) \\ f_{23}\left(H^*_{pq,j}\right) & e_{23}\left(H_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r^2 \\
& \begin{bmatrix} 0 & f_{13}\left(H^*_{i,mn}\right) \\ f_{31}\left(H^*_{pq,j}\right) & e_{31}\left(H_{pq,mn}\right) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& \qquad\qquad g=e \\
\phi_{r^2Z}(g) \;=\; & \begin{bmatrix} 0 & f_{31}\left(H^*_{i,mn}\right) \\ f_{13}\left(H^*_{pq,j}\right) & e_{13}\left(H_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r \\
& \begin{bmatrix} 0 & f_{12}\left(H^*_{i,mn}\right) \\ f_{21}\left(H^*_{pq,j}\right) & e_{21}\left(H_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r^2 \\
& \begin{bmatrix} 0 & f_{23}\left(H^*_{i,mn}\right) \\ f_{32}\left(H^*_{pq,j}\right) & e_{32}\left(H_{pq,mn}\right) \end{bmatrix}
\end{aligned}
$$

where $(H_{i,mn})$ is a 5×25 matrix, $\left(H^*_{pq,j}\right)$ is a 25×5 matrix, $H_{pq,mn}$ is a 25×25 matrix, and their elements are

$$
\begin{aligned}
H_{i,mn} &= H_m\delta_{i,n} - H_n\delta_{i,m}, \\
H_{pq,j} &= H_p\delta_{q,j} - H_q\delta_{p,j}, \\
H_{pq,mn} &= \epsilon_{pqmnk}H_k\ ,
\end{aligned}
$$

We give these fields on discrete points Z, rZ, r^2Z by the Hermitian condition $\phi_g^\dagger = R_g\phi_{g^{-1}}$, which is $\phi_Z^\dagger = R_Z\phi_Z$, $\phi_{rZ}^\dagger = R_{rZ}\phi_{r^2Z}$, $\phi_{r^2Z}^\dagger = R_{r^2Z}\phi_{rZ}$.

The other two fields ϕ_r, ϕ_{r^2} are set as

$$
\begin{aligned}
& \qquad\qquad g=e \\
\phi_r(g) \;=\; & I\begin{bmatrix} t_1\left(\sum_{i,j}\right) & \\ & s_1\left(\sum^*_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r \\
& I\begin{bmatrix} t_2\left(\sum_{i,j}\right) & \\ & s_2\left(\sum^*_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r^2 \\
& I\begin{bmatrix} t_3\left(\sum_{i,j}\right) & \\ & s_3\left(\sum^*_{pq,mn}\right) \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& \qquad\qquad g=Z \\
\phi_r(g) \;=\; & I\begin{bmatrix} t_1\left(\sum^*_{i,j}\right) & \\ & s_1\left(\sum_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=rZ \\
& I\begin{bmatrix} t_2\left(\sum^*_{i,j}\right) & \\ & s_2\left(\sum_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r^2Z \\
& I\begin{bmatrix} t_3\left(\sum^*_{i,j}\right) & \\ & s_3\left(\sum_{pq,mn}\right) \end{bmatrix},
\end{aligned}
$$

where $I=\sqrt{-1}$ and t_i, s_i are real parameters, $\Sigma_{pq,mn} = \Sigma_{p,q}\delta_{q,n} + \Sigma_{q,n}\delta_{p,m}$, and $(\Sigma_{i,j})$ is a 5×5 traceless Hermitian matrix, i.e $(\Sigma_{i,j}) = (\Sigma_{i,j})^\dagger$ and Tr $\sum$=0. The Hermitian condition $\phi_{r^2}^\dagger = R_{r^2}\phi_r$ gives the values of ϕ_{r^2} on discrete points as

$$
\begin{aligned}
& \qquad\qquad g=e \\
\phi_{r^2}(g) \;=\; & -I\begin{bmatrix} t_3\left(\sum_{i,j}\right) & \\ & s_3\left(\sum^*_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r \\
& -I\begin{bmatrix} t_1\left(\sum_{i,j}\right) & \\ & s_1\left(\sum^*_{pq,mn}\right) \end{bmatrix}; \\
& \qquad\qquad g=r^2 \\
& -I\begin{bmatrix} t_2\left(\sum_{i,j}\right) & \\ & s_2\left(\sum^*_{pq,mn}\right) \end{bmatrix}
\end{aligned}
$$

$$g = Z$$
$$\phi_{r^2}(g) \;=\; -I\begin{bmatrix} t_3\left(\sum^*_{i,j}\right) & \\ & s_3\left(\sum_{pq,mn}\right)\end{bmatrix};$$
$$g = rZ$$
$$-I\begin{bmatrix} t_1\left(\sum^*_{i,j}\right) & \\ & s_1\left(\sum_{pq,mn}\right)\end{bmatrix};$$
$$g = r^2 Z$$
$$-I\begin{bmatrix} t_2\left(\sum^*_{i,j}\right) & \\ & s_2\left(\sum_{pq,mn}\right)\end{bmatrix}.$$

Actually, in the assignments of those fields ϕ_r, ϕ_{r^2} we impose a symmetry $R_Z\phi = -\phi^*$. It is interesting to find that this constraint corresponds to the discrete symmetry introduced in the standard SU(5) grand unification model. In the SU(5) model, the Higgs potential is required to be invariant under a discrete transformation $H \to -H$, $\sum \to -\sum$, which can remove undesirable terms in the potential.

3.2 Lagrangian of the model

After taking the assignments of Yang–Mills fields and Higgs fields, we are now ready to write down the Lagrangian of the fermionic sector from (3.4),

$$\begin{aligned}\mathcal{L}_{\mathrm{F}} = &\sum_{A,k}\overline{\psi}_{k,A} i\gamma^\mu D_\mu \psi_{k,A} + \sum_{A,k,l}\overline{\chi}_{kl,A} i\gamma^\mu D_\mu \chi_{kl,A} \\ &+2\sum_{A,B,k,l} M_{1A,B}\overline{\chi^C}_{kl,A}\psi_{k,B}H^\dagger_l + \text{h.c.} \\ &+\sum_{A,B}\sum_{pqklm} M_{2A,B}\overline{\chi^C}_{pq,A}\chi_{kl,B}\epsilon_{pqklm}H_m + \text{h.c.}\end{aligned} \quad (3.6)$$

where A, B are generation indices, the other indices refer to the SU(5) group and $M_{1A,B}$ $M_{2A,B}$ are elements of matrices

$$M_1 = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33}\end{bmatrix}, \quad M_2 = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33}\end{bmatrix}.$$

It is easy to show that

$$\overline{\chi^C}_{ij,A}\chi_{kl,B}\epsilon_{ijklm}H_m = \overline{\chi^C}_{kl,B}\chi_{ij,A}\epsilon_{ijklm}H_m, \quad (3.7)$$

so we may set M_2 to be symmetric matrix, i.e $e_{AB} = e_{BA}$, or

$$M_2 = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33}\end{bmatrix}.$$

The Lagrangian of the bosonic sector may be derived from the generalized differential calculation on $M^4 \times Z_2 \times Z_3$. From the assignment of fields on discrete groups and basic knowledge of non-commutative geometry, we can write down the Lagrangian of gauge fields. For the sake of simplicity, we set $\eta_Z = \eta_{rZ} = G$ and $\eta_r = G_1$. After a quite tedious calculation, we obtain the result as follows:

$$\begin{aligned}\mathcal{L}_{\mathrm{G}} &= -\tfrac{1}{N}\langle F, \overline{F}\rangle \\ &= -\tfrac{g^2}{4N}66F_{\mu\nu}F^{\mu\nu} + \tfrac{16\beta}{N}\tfrac{G}{U^2}D_\mu H^\dagger D^\mu H \\ &\quad +\tfrac{4\alpha}{N}\tfrac{G_1}{U_1^2}Tr(D_\mu\Sigma^\dagger D^\mu\Sigma) \\ &\quad -[V(H,\Sigma) + V(\Sigma) + V(H)],\end{aligned} \quad (3.8)$$

where

$$\begin{aligned}\alpha &= t_1^2 + t_2^2 + t_3^2 + 10(s_1^2 + s_2^2 + s_3^2), \\ \beta &= Tr(2M_1M_1^\dagger + 3M_2M_2^\dagger),\end{aligned}$$

and

$$\begin{aligned}D_\mu H &= (\partial_\mu + igA_\mu)H \\ D_\mu \textstyle\sum &= \partial_\mu \textstyle\sum + ig(A_\mu \textstyle\sum - \textstyle\sum A_\mu),\end{aligned}$$

which show that Higgs fields H and $\sum$ are vector and adjoint representations of the SU(5) group, respectively. Here we write gauge bosons, Higgs fields $\sum$, and H in their matrix forms [14] as

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix} & & & X_1 & Y_1 \\ \multicolumn{3}{c}{[G\ -2B/\sqrt{30}]^\alpha{}_\beta} & X_2 & Y_2 \\ & & & X_3 & Y_3 \\ X_1^\dagger & X_2^\dagger & X_3^\dagger & W^3/\sqrt{2} + 3B/\sqrt{30} & W^\dagger \\ Y_1^\dagger & Y_2^\dagger & Y_3^\dagger & W^- & -W^3/\sqrt{2} + 3B/\sqrt{30}\end{bmatrix} \quad (1)$$

(3.9)

$$\sum = \left[\begin{array}{ccc|cc} & & & \Sigma_{X1} & \Sigma_{Y1} \\ [\Sigma_8]^\alpha{}_\beta & - & 2\Sigma_0/\sqrt{30} & \Sigma_{X2} & \Sigma_{Y2} \\ & & & \Sigma_{X3} & \Sigma_{Y3} \\ \hline \Sigma^\dagger_{X1} & \Sigma^\dagger_{X2} & \Sigma^\dagger_{X3} & \multicolumn{2}{c}{[\Sigma_3]^r_s + 3\Sigma_0/\sqrt{30}} \\ \Sigma^\dagger_{X1} & \Sigma^\dagger_{X2} & \Sigma^\dagger_{X3} & & \end{array}\right]$$

(3.10)

$$H = \begin{bmatrix} H_{t_1} \\ H_{t_2} \\ H_{t_3} \\ H_{d_1} \\ H_{d_2}\end{bmatrix}. \quad (3.11)$$

Before giving the expression of the potential, we normalize the coefficient of dynamical terms in the above Lagrangian by taking the values of the normalization constant N and metrics G, G_1 as

$$\begin{aligned} N &= 66g^2 = 16\beta\frac{G}{U^2} = 4\alpha\frac{G_1}{U_1^2} , \\ G &= \frac{33}{8}\frac{1}{\beta}g^2U^2 , \\ G_1 &= \frac{33}{2}\frac{1}{\alpha}g^2U_1^2 . \end{aligned}$$

Then the Lagrangian of the gauge fields becomes

$$\begin{aligned} \mathcal{L}_{\rm G} = & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu H^\dagger D_\mu H \\ & +{\rm Tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) \\ & -[V(H,\Sigma)+V(\Sigma)+V(H)], \end{aligned} \tag{3.12}$$

and the potential is given as

$$\begin{aligned} & V(\Sigma) = -m_1^2{\rm Tr}\Sigma^2 + \lambda_1({\rm Tr}\Sigma^2)^2 + \lambda_2{\rm Tr}\Sigma^4 , \\ & V(H) = -m_2^2 H^\dagger H + \lambda_3 (H^\dagger H)^2 \\ & V(H,\Sigma) = \lambda_4({\rm Tr}\Sigma^2)H^\dagger H + \lambda_5 H^\dagger \Sigma^2 H , \end{aligned}$$

where

$$\begin{aligned} m_1^2 &= g^2U_1^2\left(\frac{33}{2\alpha} - \frac{99}{32}\frac{\alpha}{\beta^2}\frac{U^4}{U_1^4}\right) , \\ m_2^2 &= g^2U^2\left(\frac{33}{4\beta} - 66\frac{1}{\alpha}\frac{U_1^2}{U^2}\right) , \end{aligned} \tag{3.13}$$

$$\begin{aligned} \lambda_1 &= \frac{99}{2}g^2{\rm Tr}(SS^\dagger) , \\ \lambda_2 &= \frac{33}{4}g^2{\rm Tr}(TT^\dagger + 10{\rm Tr}SS^\dagger) , \\ \lambda_3 &= \frac{33}{4}\frac{g^2}{\beta^2}{\rm Tr}\{[{\rm Diag}(M_1M_1^\dagger)]^2 + [{\rm Diag}(M_1^\dagger M_1)]^2 \\ & \quad +2[{\rm Diag}(M_2M_2^\dagger)]^2 + 2[{\rm Diag}(M_2^\dagger M_2)]^2\} , \\ \lambda_4 &= \frac{33}{8}\frac{g^2}{\beta}{\rm Tr}[{\rm Diag}(M_1^\dagger M)T + {\rm Diag}(M_1M_1^\dagger)S \\ & \quad +4{\rm Diag}(M_2M_2^\dagger)S] , \\ \lambda_5 &= \frac{33}{8}\frac{g^2}{\beta}{\rm Tr}[{\rm Diag}(M_1M_1^\dagger)S - 2{\rm Diag}(M_2M_2^\dagger)S \\ & \quad -{\rm Diag}(M_1^\dagger M)T] . \end{aligned}$$

In the above expressions, we have used the following notation:

$$T = \frac{1}{\alpha}\begin{bmatrix} t_1^2+t_3^2 & & \\ & t_1^2+t_2^2 & \\ & & t_2^2+t_3^2 \end{bmatrix}$$

$$S = \frac{1}{\alpha}\begin{bmatrix} s_1^2+s_3^2 & & \\ & s_1^2+s_2^2 & \\ & & s_2^2+s_3^2 \end{bmatrix} .$$

It is easy to show that

$${\rm Tr}(T+10S) = 2;$$

For a 3×3 matrix M, we define ${\rm Diag}(M)$ as the diagonal part of M

$${\rm Diag}(M) = \begin{bmatrix} M_{11} & & \\ & M_{22} & \\ & & M_{33} \end{bmatrix} .$$

To express the above formulas in a simple form, we redefine parameters by absorbing some constants in free parameter U and U_1:

$$\mu = \frac{33g^2U}{\beta}, \quad \mu_1 = \frac{33g^2U_1}{\alpha},$$

and

$$\begin{aligned} \hat{s}_1 &= \tfrac{s_1^2+s_3^2}{\alpha}, \quad \hat{t}_1 = \tfrac{t_1^2+t_3^2}{\alpha}, \\ \hat{s}_2 &= \tfrac{s_1^2+s_2^2}{\alpha}, \quad \hat{t}_2 = \tfrac{t_1^2+t_2^2}{\alpha} , \\ \hat{s}_3 &= \tfrac{s_2^2+s_3^2}{\alpha}, \quad \hat{t}_3 = \tfrac{t_2^2+t_3^2}{\alpha} \end{aligned} \tag{3.14}$$

where $\hat{s}_i, \hat{t}_i$ $(i = 1,2,3)$, are positive real numbers. Then we can write T, S, m_1^2 and m_2^2 in a simple form:

$$T = \begin{bmatrix} \hat{t}_1 & & \\ & \hat{t}_2 & \\ & & \hat{t}_3 \end{bmatrix} , \quad S = \begin{bmatrix} \hat{s}_1 & & \\ & \hat{s}_2 & \\ & & \hat{s}_3 \end{bmatrix} ,$$

$$m_1^2 = \mu_1^2\left(\frac{1}{2} - \frac{3}{32}\frac{\mu^4}{\mu_1^4}\right), \quad m_2^2 = \mu^2\left(\frac{1}{4} - 2\frac{\mu_1^2}{\mu^2}\right).$$

In the previous calculation, we only take into account the term $\langle F, \overline{F}\rangle$. It can be shown that in this case the Higgs potential cannot give correct symmetry breaking mechanism of the SU(5) group. Fortunately, if the term $\langle F\rangle$ is introduced in the Lagrangian, we can get correct results. It is easy to show that,

$$\langle F\rangle = \frac{16}{U^2}G\beta H^\dagger H + 4\alpha\frac{G_1}{U_1^2}{\rm Tr}(\Sigma^2).$$

Hence we should introduce the Lagrangian

$$\mathcal{L} = -\frac{1}{N}(\langle F,\overline{F}\rangle + q'\langle F\rangle).$$

Using previous formulas, one finds that

$$-\frac{q'}{N}\langle F\rangle = -q'H^\dagger H - q'{\rm Tr}(\Sigma^2).$$

If we set $q' = q\mu_1^2$ and recalculate the Lagrangian of the gauge fields, we will find that only coefficients m_1^2, m_2^2 are modified as,

$$\begin{aligned} m_1^2 &= \mu_1^2\left(\tfrac{1}{2} + q - \tfrac{3}{32}\tfrac{\mu^4}{\mu_1^4}\right) , \\ m_2^2 &= \mu^2\left[\tfrac{1}{4} + (q-2)\tfrac{\mu_1^2}{\mu^2}\right]. \end{aligned} \tag{3.15}$$

4 The realistic SU(5) model and Higgs mechanism

In the last section, we have completed the model building of generalized gauge theory on $M^4 \times Z_2 \times Z_3$, with the potential of Higgs fields derived directly from the calculation of non-commutative geometry. However, some crucial points need to be further studied, such as whether the potential provides the desired mechanism of gauge symmetry breaking, [i.e SU(5) $\longrightarrow$ SU(3) $\times$ SU(2) $\times$ $U(1)$ $\longrightarrow$ SU(3) $\times$ U(1)] and do the results suit a description of the physical phenomena?

4.1 Realistic SU(5) model

It is known that there are two mass scales in the SU(5) model: of the X,Y and of W,Z gauge bosons. There exits a vast gauge hierarchy in the SU(5) model: M_X is larger than M_W by something like 12 orders of magnitude. In this section, we show that the model we built in the last section may give rise to the desired symmetry breaking and gauge hierarchy if we impose the following condition on the parameters:

$$\begin{aligned} \mu_1 &\ll \mu \,, \\ F &= \tfrac{30\lambda_4+9\lambda_5}{60\lambda_1+14\lambda_2} < 1 \,, \\ q &= \tfrac{4+F}{2(1-F)} \,. \end{aligned} \tag{4.1}$$

Now subject to (4.1) we may write down the Bosonic part of the Lagrangian as

$$\begin{aligned} \mathcal{L}_{\rm G} = & -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu H^\dagger D_\mu H + {\rm Tr}(D_\mu \Sigma^\dagger D^\mu \Sigma) \\ & + m_1^2 {\rm Tr}\Sigma^2 - \lambda_1({\rm Tr}\Sigma^2)^2 - \lambda_2 {\rm Tr}\Sigma^4 \\ & + m_2^2 H^\dagger H - \lambda_3 (H^\dagger H)^2 - \lambda_4 {\rm Tr}\Sigma^2 H^\dagger H \\ & - \lambda_5 H^\dagger \Sigma^2, \end{aligned} \tag{4.2}$$

where

$$m_1^2 = \frac{5}{2(1-F)}\mu_1^2, \quad m_2^2 = \frac{1}{4}\mu^2 + F m_1^2,$$

$$\begin{aligned} \lambda_1 &= \tfrac{99}{2} g^2 {\rm Tr}(SS^\dagger) \,, \\ \lambda_2 &= \tfrac{33}{4} g^2 {\rm Tr}(TT^\dagger + 10{\rm Tr}SS^\dagger) \,, \\ \lambda_3 &= \tfrac{33}{4}\tfrac{g^2}{\beta^2} {\rm Tr}\{[{\rm Diag}(M_1 M_1^\dagger)]^2 + [{\rm Diag}(M_1^\dagger M_1)]^2 \\ & \quad + 2[{\rm Diag}(M_2 M_2^\dagger)]^2 + 2[{\rm Diag}(M_2^\dagger M_2)]^2\} \,, \\ \lambda_4 &= \tfrac{33}{8}\tfrac{g^2}{\beta} {\rm Tr}[{\rm Diag}(M_1^\dagger M_1)T + {\rm Diag}(M_1 M_1^\dagger)S \\ & \quad + 4{\rm Diag}(M_2 M_2^\dagger)S] \,, \\ \lambda_5 &= \tfrac{33}{8}\tfrac{g^2}{\beta} {\rm Tr}[{\rm Diag}(M_1 M_1^\dagger)S - 2{\rm Diag}(M_2 M_2^\dagger)S \\ & \quad - {\rm Diag}(M_1^\dagger M)T], \end{aligned} \tag{4.3}$$

In those expressions, matrices M_1, M_2, T, and S are defined as

$$M_1 = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}, \quad M_2 = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix},$$

$$T = \begin{bmatrix} \hat{t}_1 & & \\ & \hat{t}_2 & \\ & & \hat{t}_3 \end{bmatrix}, \quad S = \begin{bmatrix} \hat{s}_1 & & \\ & \hat{s}_2 & \\ & & \hat{s}_3 \end{bmatrix},$$

where $\hat{s}_i, \hat{t}_i$ are positive real numbers and satisfy the condition $\frac{1}{2}{\rm Tr}(T + 10S) = 1$.

So far we have constructed a realistic SU(5) model. Our next task is to see whether it gives us the desired physical results.

4.2 Symmetry breaking

Since for parameters λ_1 and λ_2 in (4.3), $\lambda_2 > 0$ and $\lambda_1 > -(7/30)\lambda_2$, potential $V(\Sigma)$ reaches its minimum at

$$\Sigma_0 = V_1 \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{bmatrix},$$

where $V_1^2 = m_1^2/(60\lambda_1 + 14\lambda_2)$, which was derived by Li [13]. For the first stage, SU(5) gauge symmetry is spontaneous broken down to SU(3) $\times$ SU(2) $\times$ U(1) as the scalar Σ develops VEV, $\langle\Sigma\rangle = \Sigma_0$. Because Σ is a scalar in the adjoint representation of SU(5), mass terms for the G^α_β, W_r, B fields remain to be zero, while the X and Y bosons acquire their masses

$$M_X = M_Y = \sqrt{\frac{25}{2}} g V_1.$$

For the second stage, gauge symmetry SU(3) $\times$ SU(2) $\times$ U(1) are broken to SU(3) $\times$ U(1) as scalar field H takes its VEV as

$$\langle H \rangle = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ V_2 \end{bmatrix},$$

where $V_2^2 = m_{\rm d}^2/\lambda_3$, $m_d^2 = m_2^2 - (30\lambda_4 + 9\lambda_5)V_1^2 = 1\,\mu^2/4$. Then bosons W and B obtain masses,

$$M_W = \frac{1}{2} g V_2, \quad M_B = \sqrt{\frac{2}{5}} g V_2.$$

Meanwhile, Higgs fields also obtain their masses in this model, their values are listed in the following table.

Scalar fields	$[\text{mass}]^2$
$[\Sigma_8]^\alpha{}_\beta$	$20\lambda_2 V_1^2$
$[\Sigma_3]^\alpha{}_\beta$	$80\lambda_2 V_1^2$
Σ_0	$4m_1^2$
H_{t_α}	$\lambda_3 V_2^2 + 5\lambda_5 V_1^2$
H_{d_r}	$\lambda_3 V_2^2$

(4.4)

It is interesting to note that

$$\frac{m_{\rm d}^2}{m_W^2} = \frac{33}{\beta^2}{\rm Tr}\{[{\rm Diag}(M_1 M_1^\dagger)]^2 + [{\rm Diag}(M_1^\dagger M_1)]^2 + 2[{\rm Diag}(M_2 M_2^\dagger)]^2 + 2[{\rm Diag}(M_2^\dagger M_2)]^2\} \quad (4.5)$$

is a quantity that depends on the fermionic mass matrix. This relation does not exist in the original SU(5) Grand Unified Model.

Because parameters μ and μ_1 were chosen to be $\mu \ll \mu_1$ in conditions (4.1), it is easy to find $V_2 \ll V_1$ in VEV, which means that the masses of gauge bosons X and Y may be as heavy as 12 orders of magnitude larger than that of gauge bosons W and B. Therefore the gauge hierarchy problem is fitting here. In fact, to realize $\mu \ll \mu_1$, we should take $U \ll U_1$ in the fermion lagrangian (3.4). From the point view of the non-commutative geometry approach, U is a parameter labeling the distance between two discrete points of Z_2, and U_1 is that labeling the distance between three discrete points of Z_3. These two geometry quantities control the mass scales of symmetry broken in our model.

5 Concluding remarks

We have constructed an SU(5) model using generalized gauge theory on $M^4 \times Z_2 \times Z_3$. We have shown that the Higgs mechanism is automatically included in the generalized gauge theory by introducing the Higgs fields as a kind of gauge fields with respect to the discrete groups and the Yukawa couplings automatically given by the generalized gauge coupling principle. Then we arrange the parameters appropriately and obtain the minimal SU(5) grand unified model. In this model, the Higgs potential can lead to the spontaneous symmetry breaking mechanism of $\mathrm{SU}(5) \longrightarrow \mathrm{SU}(3) \times \mathrm{SU}(2) \times U(1) \longrightarrow \mathrm{SU}(3) \times U(1)$, and they take place in two different gauge hierarchy scalars. There are also two scalars H and $\sum$, the vector and adjoint representations of SU(5) group to break down gauge symmetry and enable the particles to be massive. In the construction of the model, we arrange H and $\sum$ in the connection matrices ϕ_Z, ϕ_{Zr}, ϕ_{Zr^2}, ϕ_r, and ϕ_{r^2}. We want to emphasize that this assignment is unique in general. Suppose they are set in a "wrong" place, their transformation properties under the SU(5) group will not be satisfied. It is worthwhile to point out that the hierarchy scalars depend on two geometry quantities, namely, the distance of the two discrete points in Z_2 and that of three discrete points in Z_3. One of the interesting starting points of this approach is to understand the discrete groups Z_2 and Z_3 as charge conjugation transformation and generation translation in the free fermion Lagrangian, although they are broken after the arrangements of gauge fields. This is completely different from the previous work.

There exist some differences between the parameters of the reconstructed model and the standard SU(5) grand unified model. In the standard SU(5) model, there are the following free parameters:

g, the SU(5) coupling constant,
M_1, M_2, the mass matrices,
$m_1^2, m_2^2, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, the parameters in the potential.

In our reconstructed model, coupling constant g, mass matrices M_1, M_2, and m_1, m_2 are also free parameters. Instead of parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, we introduced two matrices S and T and they satisfy the condition $\frac{1}{2}{\rm Tr}(T + 10S) = 1$. Observe that the number of parameters is equal in these two models, but now parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are functions of M_1, M_2, S, T in the reconstructed model, so they are not as free as in the standard SU(5) model. One result of this property is that the ratio of M_{H_d}/M_W is a function of mass matrices, which means a complex relation exists among the masses of particles at tree level. Therefore, it needs to be studied further whether there are more relations. This approach may also be used to study more models like the left–right symmetry model, the SO(10) grand unified model, and the supersymmetry model.

Acknowledgements. This work is supported in part by the Chinese Postdoctoral Foundation and National Science Foundation. The authors would like to thank Prof. H-Y Guo, K. Wu and Z.Y. Zhao for helpful discussions and Dr. C. Liu and Y.K. Lau for useful comments.

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